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MATHEMATICAL MODEL OF NETWORKS WITH SHARED RESOURCES

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In this paper, a mathematical model is studied of a network integrating distributed resources and users. A Probability Theory approach to modeling shared utilization of distributed resources is taken based on the Limiting Theorems technique. This work has successfully resolved the problem of obtaining an asymptotic estimate of bandwidth load on data channels and hardware load on resource carriers, setting from various assumptions in respect of probability distributions of user demand within fixed time intervals.

Results and conclusions of this work have a great significance for information technology applications, and this work will be found of interest by researchers in both theoretical and practical aspects of Probability Theory applications in information technologies.

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В работе рассматривается математическая модель сети распределённых ресурсов и потребителей. Применён теоретико-вероятностный метод моделирования совместного использования распределённых сетевых ресурсов, основанный на аппарате предельных теорем теории вероятности. В работе успешно решена задача асимпотической оценки величины нагрузки на информационные каналы и носители ресурсов, исходя из различных изначальных предположений о вероятности распределения спроса индивидуальных пользователей ресурсов в течение фиксированных периодов времени.

Результаты работы имеют большое значение для различныхинформационных приложений и представляют интерес для научных работников, занятых как теоретическими, так и практическими аспектами теоретико-вероятностных методов в информатике.

Chapter I.

Introduction.

Various industrial applications utilise networked resources such as computers joined into local or wide area networks (LAN and WAN), communications systems, control systems based on industrial controllers and servers connected by means copper wire, fibre-optic or wireless communication links. Such networks are sensitive parts of real-time control/monitoring systems that must be able to provide immediate reaction to some external events and or/generate responses (e.g., facility shutdown, trip signals, alarms, etc.) Real Time systems divide into Soft Real Time and Hard Real Time systems, meaning that Hard Real Time Systems are required to provide a response to an external event within a determined period of time, whether short or not, whereas in Soft Real Time systems, short but predictable delays of undetermined but estimable duration are tolerated. A Real Time application's responsiveness is directly related to the architecture of the operating system that runs it, and communication bottlenecks – i.e., shared resources that can be simultaneously requested access to by more processes or threads than can actually gain access to the resources at a time. Typically, integrity aspects of the shared resource problem are resolved through the use semaphores, mutexes and events supported by most of programming environments. However, blocking access to a shared resource for a process (thread) obviously means that the process (thread) will have to be placed in a queue and therefore suspended. Of course, suspensions may adversely affect the system's responsiveness -- and in fact are the principal reason for sluggishness of some practical application software. In view of the important place that resource sharing has in the performance of networked systems, it is critical that the efficiency of resource sharing be estimated prior to design phase to gauge the overall system performance (i.e., the expected average and maximal loads on the bottlenecks, longest queues, expected queue length, expected resource release wait time, etc.)

In this work, a mathematical model of a shared resource network has been developed, and efficiency estimation technique proposed that can be used in performance calibration of both real time software and hardware, as well as conceptual design of server/client systems in general. The proposed technique is based on the utilisation of Probability Theory Limiting Theorems and allows for universal application.

Techniques and methods of this work have been utilised in performance tuning of a commercial Soft Real Time System – Time Card Registration Software, LAN-based server/client application for maintaining employee databases and worked time reports on a multiple project basis. The source code of the computer programme for Windows NT that provides the main functionality of the application's client component as well as sample interfaces are supplied in Chapter IV.

Chapter I contains some basic definitions and formulation of the problems that are solved later in the work. Chapter II presents detailed description of a shared resource network and provides solutions to previously posed problems. Chapter III demonstrates an application of the model from Chapter II in the design of a particular Soft Real Time System. Chapter IV contains source code of the system's core software component and user interfaces.

Chapter II.

Basic Definitions.

A typical network that contains a resource shared among several network clients is shown schematically in Fig. 1. Straight lines fanned around the resource represent client's accesses to the resource. Most shared resources impose a restriction on the number of clients that can access the resource at a time. Such restriction may be related to software aspects of multitasking or hardware limitations. E.g., the contents of a shared memory block or disk storage area may be corrupted if simultaneously accessed by two or more processes or threads or cause an unstable behaviour of a thread that had gained access to the shared resource before another thread completed modifying its contents.

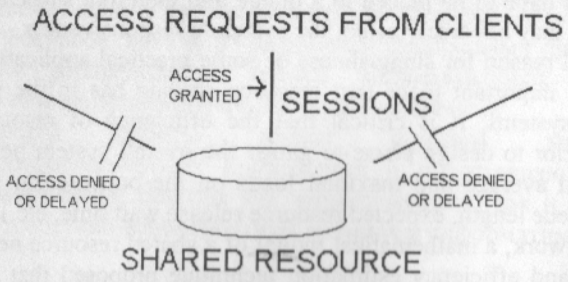

Fig 1. Schematic of a network with a shared resource.

Examples of sharing problems related to hardware limitations can be found in abundance in telecommunications: communications networks usually have a hub that can service only a specific number of clients at a time. In order to overcome such problems, special techniques can be utilised for safeguarding shared resources while in use from any further access until the client that

currently uses the resource releases it. However, such techniques (e.g., utilisation of semaphores and mutexes) result in the overall performance degradation. It is therefore vitally important that we can estimate the total load on the shared resources prior to incorporating networking principles into industrial applications, especially Real Time Systems.

Hereinafter in this work, *session* means a client's use of a shared resource from *log-on* to *log-off*. Typically, session length is a random variable and so we use probability density function $p(x), -\infty < x < +\infty$ that determines the probability of a session length's being within a range of values. I.e.,

$$\mathbf{P}(a \le \xi < b) = \int_a^b p(x)dx$$

is the probability that a session duration will be longer than or at least equal to a and strictly shorter than b (the measurement unit does not matter and in practical applications depends on the type of modelled resource – e.g., can be milliseconds for Ethernet or Token-ring LAN's, minutes for large-scale communications hubs and nanoseconds for shared memory resources). Figures 3 and 5 show examples of probability densities that can be used in practice – later in this work we will provide reasons for using Poisson's and Normal distributions.

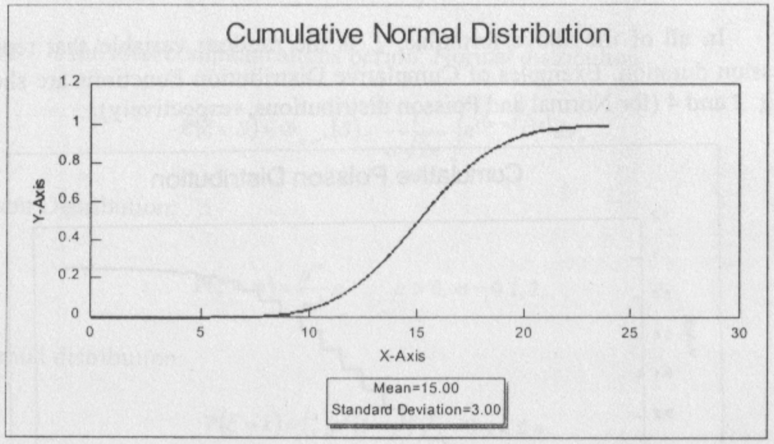

Fig. 2. Cumulative Normal Distribution.

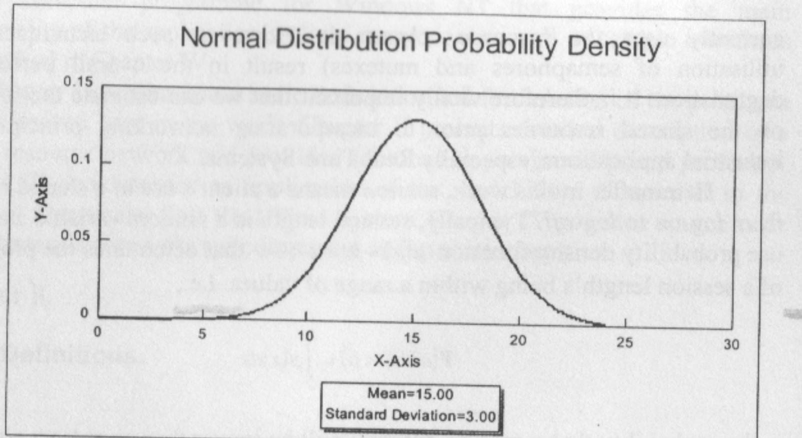

Fig. 3. Normal Distribution Probability Density.

Equating a in the above formula to zero, we will obtain the so-called Cumulative Probability Distribution function:

$$F_\xi(x) = \mathbf{P}(\xi < x) = \mathbf{P}(0 \le \xi < x) = \int_0^x p(x)dx$$

In all of the above formulae, ξ is the random variable that represents session duration. Examples of Cumulative Distribution Functions are shown in Fig. 2 and 4 (for Normal and Poisson distributions, respectively).

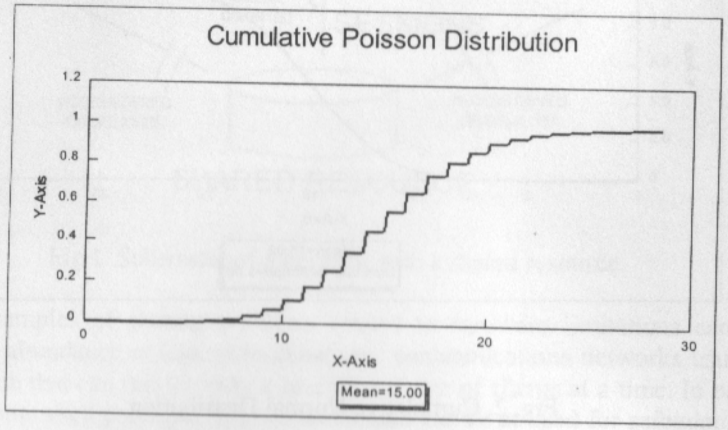

Fig. 4. Cumulative Poisson Distribution.

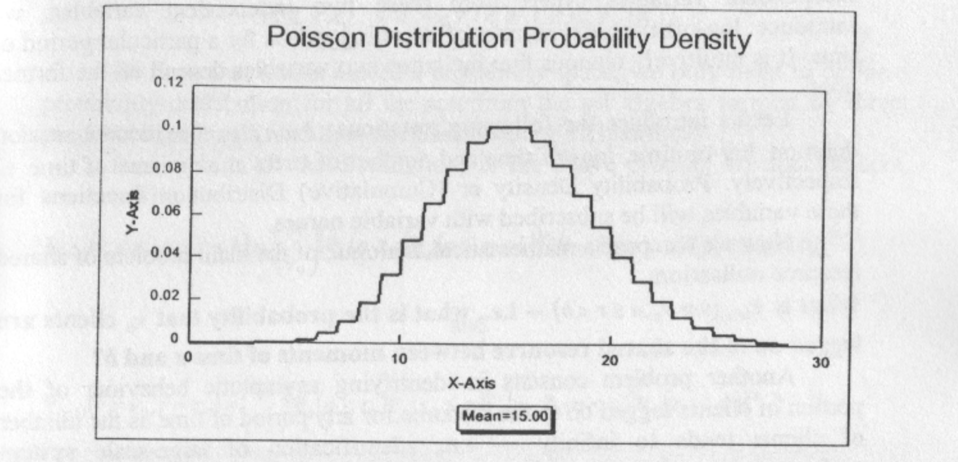

Fig. 5. Poisson Distribution Probability Density.

In addition to the random variable of session duration, we can also introduce another random quantity: the log-on time. We can study various distributions for the log-on time: uniform distribution

$$p_2(x) = \frac{1}{T}$$

where T is the total communications period; Normal distribution

$$\mathbf{P}(\xi \in S) = \mathbf{\Phi}_{a,\sigma^2}(S) = \frac{1}{\sigma\sqrt{2\pi}} \int_S e^{-(u-a)^2/2\sigma^2} du\,,$$

Poisson Distribution:

$$\mathbf{P}(\xi = m) = \frac{\mu^m}{m!} e^{-\mu}, \quad \mu > 0, m = 0, 1, 2, \ldots,$$

Binomial distribution:

$$\mathbf{P}(\xi = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le k \le n.$$

Exponential Distribution:

$$\mathbf{P}(\xi < x) = F(x) = \begin{cases} 1 - e^{-\alpha x} & \text{if} \quad x \ge 0, \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Note that session duration and log-on time are, generally speaking, independent variables. Apart from these two independent variables, we introduce: log-off time and number of users logged-on for a particular period of time. It is intuitively obvious that the latter two variables depend on the former ones.

Let us introduce the following notations: ξ, η, χ, (ν,τ) denote session duration, log-on time, log-off time and number of users at a moment of time τ, respectively. Probability Density or (Cumulative) Distribution Functions for these variables will be subscribed with variable names.

Now we can pose a mathematical analogue of the main problem of shared resource utilisation:

What is $\mathbf{P}_{(\nu,\tau)}(\nu=\nu_0, a\le\tau<b)$ – i.e., what is the probability that ν_0 clients are logged on to the shared resource between moments of time *a* and *b*?

Another problem consists in identifying asymptotic behaviour of the portion of clients logged on to the resource for any period of time as the number of clients tends to infinity — i.e., identification of large-scale system performance.

Chapter III.

Mathematical Model of a Shared Resource Network.

We can easily see that $\chi=\eta+\xi$. First of all, let us calculate the probability distribution function $\mathbf{P}_\chi(\chi<b)$. We have:

$$\mathbf{P}_\chi(\chi<b)=\operatorname{mes}\chi^{-1}([0,b))=\operatorname{mes}\left\{\bigcup_{\omega\in\Omega_\eta\cap\eta^{-1}[0,b)}\xi^{-1}[0,b-\eta(\omega))\right\}=$$

$$=\operatorname{mes}\left\{\bigcup_{\omega\in\eta^{-1}[0,b)}\xi^{-1}[0,b-\eta(\omega))\right\}=\lim_{N\to\infty}\sum_{n=0}^{N}\operatorname{mes}\{\eta^{-1}[b_n,b_{n+1})\}\operatorname{mes}\{\xi^{-1}[0,b-b_n)\}=$$

$$=\int_0^b\mathbf{P}_\xi(0\le\xi<b-y)\mathbf{P}_\eta(y\le\eta<y+dy)=\int_0^b F_\xi(b-y)p_\eta(y)dy.$$

Assuming now that M clients can access the shared resource and that session duration and log-on time probability distribution functions are the same for all the clients, we can construct the probability space $\langle\Omega,\widetilde{A},\mathbf{P}\rangle$ for the whole system:

$$\Omega=\Omega_1\times\ldots\times\Omega_M,$$

$$\mathbf{P}(A_1\times A_2\ldots\times A_M)=\mathbf{P}_1(A_1)\mathbf{P}_2(A_2)\ldots\mathbf{P}_M(A_M).$$

To show that this is indeed a probability space, we only need to define a probability distribution for all the sets from the set algebra formed by direct products of intervals, and then use Caratheodory's Theorem.

Inasmuch as all the distributions in the above product are independent, we can write:

$$\mathbf{P}_{(\nu,\tau)}(\nu=\nu_0,a\le\tau<b)=\binom{M}{\nu_0}(\mathbf{P}_\eta(\eta\ge a)\mathbf{P}_\chi(a\le\chi<b))^{\nu_0}(1-\mathbf{P}_\eta(\eta\ge a)\mathbf{P}_\chi(a\le\chi<b))^{M-\nu_0}$$

and

$$\mathbf{P}_{(\nu,\tau)}(\nu=\nu_0,\tau=\tau_0)=\binom{M}{\nu_0}(\mathbf{P}_\eta(\eta<\tau_0)\mathbf{P}_\chi(\tau_0\le\chi))^{\nu_0}(1-\mathbf{P}_\eta(\eta<\tau_0)\mathbf{P}_\chi(\tau_0\le\chi))^{M-\nu_0}$$

Hence, we have obtained binomial distributions for the number of logged-on users within any given period of time and at any given moment of time, respectively – the first task has been successfully resolved.

And now we are prepared to study the asymptotic behaviour of the above probability distributions (and the corresponding Mean) when $\nu_0\to\infty, M-\nu_0\to\infty$.

We are going to use the so-called Local Limiting Theorem:

If

$$\mathrm{P}(S_n=k)=\tbinom{n}{k}p^kq^{n-k},\quad q=1-p$$

and

$$\mathrm{H}(x)=x\ln\frac{x}{p}+(1-x)\ln\frac{1-x}{1-p},\quad \widetilde{p}=\frac{k}{n}$$

then $\mathrm{P}(S_n=k)=\mathrm{P}\left(\frac{S_n}{n}=\widetilde{p}\right)\sim\frac{1}{\sqrt{2\pi n\widetilde{p}(1-\widetilde{p})}}\exp\{-n\,\mathrm{H}(\widetilde{p})\},\ k\to\infty,\ n-k\to\infty.$

In our case, we obtained:

$$\mathbf{P}_{(\nu,\tau)}\left(\frac{\nu}{M}=\widetilde{p}, a \le \tau < b\right) = \sim \frac{1}{\sqrt{2\pi M \widetilde{p}(1-\widetilde{p})}} \exp\{-M\,\mathrm{H}(\widetilde{p})\},\ \nu_0 \to \infty,\ M-\nu_0 \to \infty,$$

$$p = \mathbf{P}_\eta(\eta \ge a)\mathbf{P}_\chi(a \le \chi < b)$$

or

$$\mathbf{P}_{(\nu,\tau)}\left(\frac{\nu}{M}=\widetilde{p}, \tau=\tau_0\right) = \sim \frac{1}{\sqrt{2\pi M \widetilde{p}(1-\widetilde{p})}} \exp\{-M\,\mathrm{H}(\widetilde{p})\},\ \nu_0 \to \infty,\ M-\nu_0 \to \infty,$$

$$p = \mathbf{P}_\eta(\eta < \tau_0)\mathbf{P}_\chi(\tau_0 \le \chi).$$

(p figures in H.)

Applications to the Design of Real Time Systems.

Network-based real-time database management systems present a great potential for the application of resource sharing server/client models. Such systems may be either Soft Real Time or Hard Real Time, depending on whether they are intended for use in process control and monitoring or interactive interface with delay-tolerant clients. In any event, the technique described in the previous chapter comes in very useful for identifying potential bottlenecks and high-level loads on the system resources.

The Mathematical Model described in Chapter II perfectly suits LAN-based Database systems: by the shared resource in this case we mean a common database (or shared directory) stored on a server, sessions mean database read/write operations, logging on and off mean opening and closing the database, respectively (not logging on to and off LAN.)

As two processes may require access to the same area of the database at the same time, there will clearly be a delay in accessing the area by whichever process requests the access last. Consequently, we confront the resource-sharing problem consisting in identification of expected demand on the resource pool (i.e., the database).

The application described in the following chapter, the Time Card Registration System, implements a server/client model of database management and provides an example of resource sharing. In addition to its functionality as a

database management system, the application also provides a network-browsing feature – i.e., can be used for modelling access to all possible shared resources, not only the database.

From the results of Chapter II, we obtain a very useful formula for the Mathematical expectation

$$\mathbf{M}(\nu / a \le \tau < b) = \sum_{\nu_0}^{M} \nu_0 \binom{M}{\nu_0} (p)^{\nu_0} (1-p)^{M-\nu_0},$$

$$p = \mathbf{P}_\eta(\eta \ge a)\mathbf{P}_\chi(a \le \chi < b)$$

or

$$\mathbf{M}(\nu / \tau = \tau_0) = \sum_{\nu_0}^{M} \nu_0 \binom{M}{\nu_0} (p)^{\nu_0} (1-p)^{M-\nu_0},$$

$$p = \mathbf{P}_\eta(\eta < \tau_0)\mathbf{P}_\chi(\tau_0 \le \chi)$$

which was used in estimating the system's efficiency on Ethernet LAN.

Another useful technique in identifying asymptotic behaviour of multi-client systems consists in passing the number of clients to infinity and single session duration to zero. In a particular case when single session duration is expected to decrease like $1/\sqrt{n}$ where n is the total number of clients, we can use the following statement for estimating the aggregate session duration for all the clients.

If ξ_k, $k = 1,\dots,n$ are independent random variables such that

$$\mathbf{M}\xi_n = 0$$

$$\mathbf{D}\xi_n = 1$$

$$\mu = \mathbf{M}|\xi_k|^3 < \infty,$$

Then the variable

$$\mathrm{S}_n = \sum_1^n \xi_k, \quad \zeta_n = \mathrm{S}_n/\sqrt{n}$$

satisfies the following estimate:

$$\Delta_n = \sup_x |\mathrm{P}(\zeta < x) - \Phi(x)| < c\mu/\sqrt{n}$$

$$c \equiv const \text{ --}$$

the probability distribution for the total session duration converges to the Normal distribution.

Chapter IV.

Sample Application Source Code and Interfaces.

Shown in Fig. 6-8 are main interfaces of the Time Card Registration System.

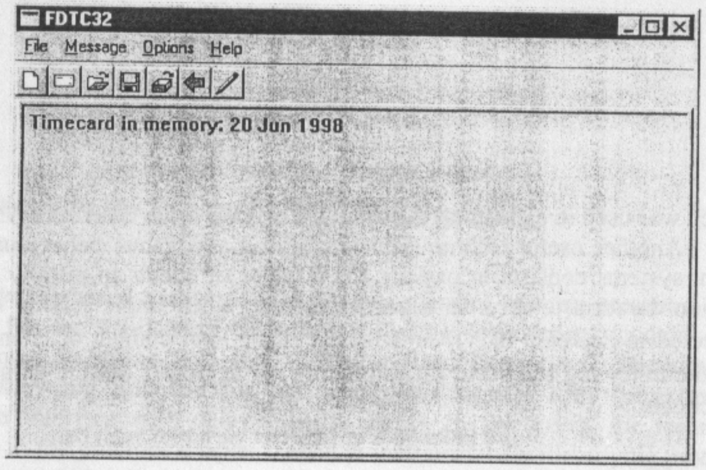

Fig. 6. Main Window.

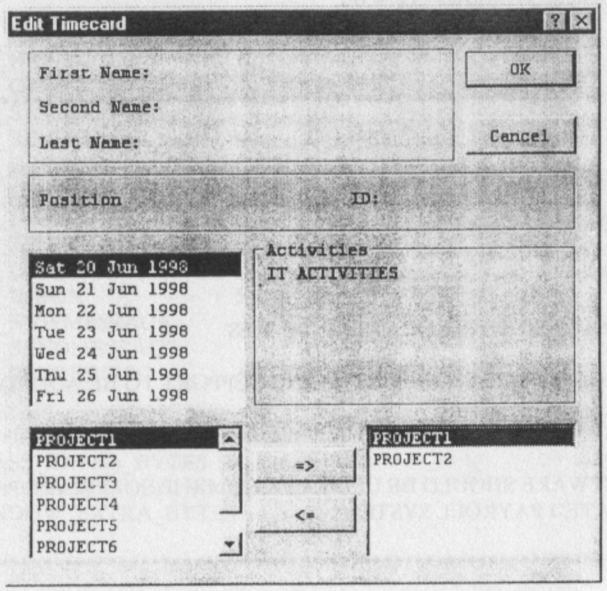

Fig. 7. Time Card Edit Dialogue

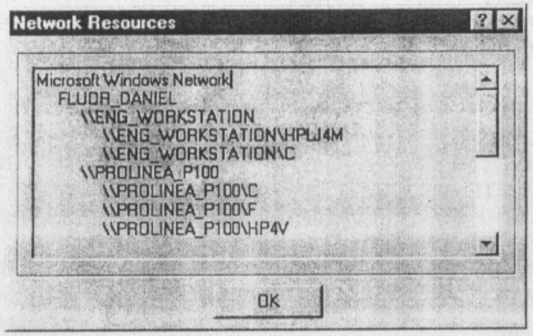

Fig. 9. Network Resource Browsing Dialogue.